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Quantum Correction to the Entropy of the (2+1)-Dimensional Black Hole

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Abstract: *The thermodinamic properties of the (2+1)-dimensional non-rotating black hole of Bañados, Teitelboim and Zanelli are discussed. The first quantum correction to the Bekenstein-Hawking entropy is evaluated within the on-shell Euclidean formalism, making use of the related Chern-Simons representation of the 3-dimensional gravity. Horizon and ultraviolet divergences in the quantum correction are dealt with a renormalization of the Newton constant. It is argued that the quantum correction due to the gravitational field shrinks the effective radius of a hole and becomes more and more important as soon as the evaporation process goes on, while the area law is not violated.*

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1 Introduction

It is well known that we do not have yet at disposal a consistent and complete 4-dimensional quantum gravity, but nevertheless a large number of interesting issues have been investigated, mainly within the semiclassical approximation. One of the most important issue is related to the black hole physics and deals with the origin of the entropy, its quantum corrections, the information loss paradox, the validity of the area law (see, for example Ref. [1]). However, it is well known that in (3+1)-dimensions, the black hole quantum physics needs several approximations.

Recently 3-dimensional gravity has been studied in detail. Despite the simplicity of the 3-dimensional case (no propagating gravitons), it is a common believe that it deserves attention as useful laboratory. In fact with surprise a black hole solution has been found by Bañados, Teitelboim and Zanelli [2], the so called BTZ black hole. In particular, a simple geometrical structure of a black hole allows exact computations since its Euclidean counterpart is locally isomorphic to the constant curvature 3-dimensional hyperbolic space H^3.

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In this paper we shall compute the first quantum correction to the semiclassical Bekenstein-Hawking entropy for the BTZ black hole due to the one-loop gravitational fluctuations, in an attempt to elucidate the statistical origin of the black hole entropy [3, 4, 5, 6] and to explore the possible relevance of the quantum fluctuations during the late stage of the black hole evaporation process.

With regard to this issues, we recall that a lot of papers have been appeared where the quantum entropy of matter fields, propagating in a black hole background, has been evaluated by means of several different techniques (see, for example, Refs. [7, 9, 10, 8, 11, 12, 13, 14, 15, 16, 17] and reference therein). We would like to stress here that we shall compute the one-loop contribution due to the quantization of the gravitational field itself. The tree level approximation to the partition function as been discussed at length in [18], using the Brown and York approach to quasi-local thermodynamic for asymptotically anti-de Sitter black holes. It is found that in $(2+1)$ dimensions, there is a thermodynamically stable black hole solution, and no negative heat capacity instantons. Thus one expects that first quantum corrections be well defined.

As far as the computation of these corrections is concerned, some work has been done in [19, 20] and a motivation of our paper is to present a detailed and possibly complete discussion on this point.

The first quantum correction of the BTZ black hole will be evaluated making use of the related Chern-Simons representation of the 3-dimensional gravity [21, 22]. It should be stressed that within this approach a preliminary statistical mechanics explanation of the Bekenstein-Hawking entropy, counting boundary states at the horizon, has been given in Ref. [23].

The contents of the paper are the following. In Sect. 2 we briefly review a geometry of the Eclidean BTZ black hole. In Sect. 3 we present a derivation of the Selberg trace formula, starting from an elementary derivation of the heat- kernel trace related to the Laplace operator, necessary for our regularization. In Sect. 4 a computation of the first quantum correction to the entropy is outlined. The paper ends with some concluding remarks in Sect. 5. In the Appendix some explicit computations are included.

2 The Euclidean BTZ Black Hole

Following [19] we summarize here geometrical aspects of the non-rotating BTZ black hole [2] which are relevant for our discussion. In the coordinates (t, r, ϕ), the static Lorentzian metric reads (8G=1 is assumed for the moment, thus the mass is dimensionless)

$$ds_L^2 = -\left(\frac{r^2}{\sigma^2} - M\right) dt^2 + \left(\frac{r^2}{\sigma^2} - M\right)^{-1} dr^2 + r^2 d\phi^2 \,, \qquad (2.1)$$

where M is the standard ADM mass and σ is a dimensional constant. A direct calculation shows that the above metric is a solution of the 3-dimensional vacuum Einstein equation with negative cosmological constant, i.e.

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu} \,, \qquad R = 6\Lambda = -\frac{6}{\sigma^2} \,. \qquad (2.2)$$

Thus, the sectional curvature k is constant and negative, namely $k = \Lambda = -1/\sigma^2$. The metric (2.1) has an horizon radius given by

$$r_+ = \sqrt{M}\sigma \,, \qquad (2.3)$$

and it describes a space-time locally isometric to the anti-de Sitter space.

The Euclidean section is obtained by the Wick rotation $t \to i\tau$ and reads

$$ds^2 = \left(\frac{r^2}{\sigma^2} - M\right) d\tau^2 + \left(\frac{r^2}{\sigma^2} - M\right)^{-1} dr^2 + r^2 d\phi^2 \,. \tag{2.4}$$

Changing the coordinates $(\tau, r, \phi) \to (y, x_1, x_2)$ by means of

$$\begin{aligned} y &= \frac{r_+}{r} e^{\frac{r_+}{\sigma}\phi} \,, \\ x_1 + i x_2 &= \frac{1}{r}\sqrt{r^2 - r_+^2} \exp\left(i\frac{r_+}{\sigma^2}\tau + \frac{r_+}{\sigma}\phi\right) \,, \end{aligned} \tag{2.5}$$

the metric becomes the one of upper-half space representation of H^3, i.e.

$$ds^2 = \frac{\sigma^2}{y^2}\left(d^2 y + dx_1^2 + dx_2^2\right) \,. \tag{2.6}$$

As a consequence, the metric Eq. (2.4) describes a manifold homeomorphic to the hyperbolic space H^3.

It is known that the group of isometries of H^3 is $SL(2, \mathbb{C})$. We shall consider a discrete subgroup $\Gamma \subset PSL(2, \mathbb{C}) \equiv SL(2, \mathbb{C})/\{\pm Id\}$ (Id is the identity element), which acts discontinuously at the point z belonging to the extended complex plane $\mathbb{C} \bigcup \{\infty\}$. We recall that a transformation $\gamma \neq Id$, $\gamma \in \Gamma$, with

$$\gamma z = \frac{az + b}{cz + d} \,, \qquad ad - bc = 1 \,, \qquad a, b, c, d \in \mathbb{C}, \tag{2.7}$$

is called elliptic if $(\mathrm{Tr}\,\gamma)^2 = (a + d)^2$ satisfies $0 \leq (\mathrm{Tr}\,\gamma)^2 < 4$, hyperbolic if $(\mathrm{Tr}\,\gamma)^2 > 4$, parabolic if $(\mathrm{Tr}\,\gamma)^2 = 4$ and loxodromic if $(\mathrm{Tr}\,\gamma)^2 \in \mathbb{C} \setminus [0, 4]$. The element $\gamma \in SL(2, \mathbb{C})$ acts on $z = (y, w) \in H^3$, $w = x_1 + i x_2$ by means of the following linear-fractional transformation:

$$\gamma z = \left(\frac{y}{|cw + d|^2 + |c|^2 y^2} \,, \frac{(aw + b)(\bar{c}\bar{w} + \bar{d}) + a\bar{c}y^2}{|cw + d|^2 + |c|^2 y^2}\right) \,. \tag{2.8}$$

The periodicity of the angular coordinate ϕ allows to describe the BTZ black hole manifold as the quotient $\mathcal{H}^3 \equiv H^3/\Gamma$, Γ being a discrete group of isometry possessing a primitive element $\gamma_h \in \Gamma$ defined by the identification

$$\gamma_h(y, w) = (e^{\frac{2\pi r_+}{\sigma}} y, e^{\frac{2\pi r_+}{\sigma}} w) \sim (y, w) \,. \tag{2.9}$$

According to the Eq. (2.8) this corresponds to the matrix

$$\gamma_h = \begin{pmatrix} e^{\frac{r_+}{\sigma}} & 0 \\ 0 & e^{-\frac{r_+}{\sigma}} \end{pmatrix} \,, \tag{2.10}$$

namely to an hyperbolic element ($\mathrm{tr}\,\gamma_h > 2$) consisting in a pure dilatation. Furthermore, since in the Euclidean coordinates τ becomes an angular type variable with period β, one is leads also to the identification

$$\gamma_e(y, w) = (y, e^{\frac{i\beta r_+}{\sigma^2}} w) \sim (y, w) \,. \tag{2.11}$$

This identification is generated by an elliptic element in the group Γ

$$\gamma_e = \begin{pmatrix} e^{i\beta\frac{r_+}{\sigma^2}} & 0 \\ 0 & e^{-i\beta\frac{r_+}{\sigma^2}} \end{pmatrix} \,, \tag{2.12}$$

as soon as $\operatorname{tr}\gamma_e < 2$, and a conical singularity will be present. If

$$\beta\frac{r_+}{\sigma^2} = 2\pi\ , \tag{2.13}$$

then $\gamma_e = Id$ and the conical singularity is absent. As a result the period is determined to be

$$\beta_H = 2\pi\frac{\sigma^2}{r_+}\ , \tag{2.14}$$

and this is interpreted as the inverse of the Hawking temperature [5]. Therefore the on-shell BTZ black hole can be regarded as a strictly hyperbolic non-compact manifold $\mathcal{H}^3$. The mass as a function of the black hole temperature $T = \beta_H^{-1}$ reads

$$M = 4\pi^2\sigma^2 T^2, \tag{2.15}$$

which shows the stability condition $\partial M/\partial T > 0$ is fulfilled. The tree-level Bekenstein-Hawking entropy S_H may be simply obtained making use of the relation

$$\beta_H = \frac{\partial S_H}{\partial M}\ . \tag{2.16}$$

Thus one has

$$S_H = 4\pi r_+ = 2A\ , \tag{2.17}$$

which is the well known "area law" for the black hole entropy. Note that $A = 2\pi r_+$ is the perimeter of the horizon. If we choose $G = 1$ instead of $8G = 1$, the entropy becomes $A/4$, as it is more familiar to a black hole physicists.

Another important thermodynamics input is the off-shell Euclidean action of a black hole, namely the action evaluated at $\beta \neq \beta_H$ (c. f. [18] for the quasi-local formalism of thermodynamics)

$$I = -\frac{1}{2\pi}\int_{\mathcal{M}}(\mathcal{R} - 2\Lambda)\sqrt{g}\, d^3x - \frac{1}{\pi}\int_{\partial\mathcal{M}}\mathcal{K}\sqrt{h}\, d^2x. \tag{2.18}$$

The boundary $\partial\mathcal{M} = S^1 \otimes S^1$ (a torus) is identified with period β (the first circle) at some fixed radius $r = R$ (the second circle), which will be taken to infinity at the end, and $\mathcal{K}$ is the trace of the extrinsic curvature of the boundary. The Euclidean action (2.18) is a divergent function of the boundary location, and therefore it is necessary to subtract the action of a chosen background [24] from it. This will be the zero mass solution, i.e. the $M = 0$ line element

$$ds_0^2 = \frac{r^2}{\sigma^2}d\tau^2 + \frac{\sigma^2}{r^2}dr^2 + r^2 d\phi^2, \tag{2.19}$$

which corresponds also to the zero temperature state; all quantities referring to this reference background have a subscript "0". As mentioned above, at $\beta \neq \beta_H$ there will be a conical singularity whose contribution to the action, as is well known, is given by the Gauss-Bonnet theorem for a disk [25], and is

$$\frac{1}{2\pi}\int_{\mathcal{M}}\mathcal{R}\sqrt{g}\, d^3x = \frac{2A}{\beta_H}(\beta_H - \beta). \tag{2.20}$$

Contribution related to a background is vanishes, since $A_0 = 0$. The difference of the actions can be computed by matching the coordinate of the boundary location in the background, $r = R_0$, to the coordinate of the boundary location in the black hole metric, $r = R$, so that the two

metrics asymptotically agree. Finally, the surface contribution is seen to vanish. Therefore, the off-shell Euclidean action becomes

$$I=-\frac{2A}{\beta_H}(\beta_H-\beta)-\frac{r_+^2\beta}{\sigma^2}=M\beta-2A, \tag{2.21}$$

where (M,β) are now independent thermodynamics variables and $r_+=\sigma\sqrt{M}$. On-shell we have $\beta=\beta_H$ and $I=-2\pi r_+$. If one identifies I with $-\ln Z$ [5], the partition function of the black hole, then the mean energy in the canonical ensemble will be

$$<E>=-\partial_\beta \ln Z = M, \tag{2.22}$$

as it was to be expected, and the entropy will be again $S=2A=4\pi r_+=4\pi\sigma\sqrt{M}$. Because $S\sim\sqrt{M}$, the partition function as a "sum over states" in semiclassical quantum $2+1$-gravity will converge, and the canonical ensemble for a black hole in equilibrium with thermal radiation will lead to a stable thermodynamics.

We conclude this section with a comment on the global geometry of the ground state. Looking at Eq. (2.19), it is clear that τ can be identified to any period β (in particular $\beta=\infty$), and that ϕ has the period 2π. Changing the coordinates as $r=\sigma^2/y$, $\tau=x_1$ and $\phi=x_2/\sigma$, one gets the metric of hyperbolic space

$$ds_0^2=\sigma^2\,\frac{dx_1^2+dx_2^2+dy^2}{y^2}, \tag{2.23}$$

and the identification $\gamma_p(w,y)=(w+\beta+i2\pi\sigma\, y)\simeq(w,y)$. This identification is generated by elements of Γ of the form

$$\gamma_{p_1}=\left(\begin{array}{cc}1&\beta\\0&1\end{array}\right), \qquad \gamma_{p_2}=\left(\begin{array}{cc}1&2\pi i\sigma\\0&1\end{array}\right), \tag{2.24}$$

which are parabolic. Thus, our reference manifold can be regarded as the quotient $\mathcal{H}_0=H^3/\Gamma_0$, where a subgroup Γ_0 has primitive parabolic elements γ_{p_1} and γ_{p_2}.

We note that for negative mass, one gets solutions with naked conical singularity [26] unless one arrives at $M=-1$, namely H^3, the Euclidean counterpart of the 3-dimensional anti-de Sitter space-time. This solution is a permissible solution and can regarded as a "bound state" [2].

3 Trace Formula and the Spectral Zeta Function

In this section we investigate the spectral properties associated with the Laplace type operator acting in a non-compact hyperbolic manifold $\mathcal{H}^3$. To go further, it is convenient to introduce spherical hyperbolic coordinates

$$y=\cos\theta\,, \qquad w=x_1+ix_2=\rho\sin\theta e^{i\varphi}\,. \tag{3.1}$$

It is easy to show that the fundamental domain of $\mathcal{H}^3$ is non-compact and it can be given as follows [19]

$$F=\{1\le\rho\le N, 0\le\theta<\pi/2, 0<\varphi<2\pi\}\ , \tag{3.2}$$

where $\ln N=2\pi r_+/\sigma$. Note that $z'=\gamma_h z=Nz$ and the corresponding transformation law for a scalar field Φ reads $\Phi(\gamma z)=\chi\Phi(z), \gamma\in\Gamma$, where χ is a finite-dimensional unitary representation (a character) of Γ.

Let us consider an arbitrary integral operator which is given by a kernel $k(z,z')$. The operator is invariant (i.e. the operator commutes with all operators of the quasi-regular representation of the group $PSL(2,\mathbb{C})$ in the space $C_0^\infty(H^3)$) if its kernel satisfy the condition $k(\gamma z,\gamma z')=k(z,z')$ for any $z,z'\in H^3$. Thus, the kernel of the invariant operator, for example the Laplace operator, is a function of the geodesic distance between z and z', namely

$$d(z,z')=\cosh^{-1}\left[1+\frac{(y-y')^2+(x_1-x_1')^2+(x_2-x_2')^2}{2yy'}\right]. \tag{3.3}$$

Then the geodesic length between the point z and $z'=\gamma_h z$ is

$$l_0=\inf d(z,\gamma_h z)=\ln N=2\pi\frac{r_+}{\sigma}. \tag{3.4}$$

It is convenient to replace such a distance with the fundamental invariant of a pair of points

$$u(z,z')=\frac{1}{2}\left(\cosh d(z,z')-1\right), \qquad u(z,z)=0, \tag{3.5}$$

and therefore $k(z,z')=k(u(z,z'))$. Finally for the sake of simplicity we put $\sigma=1$ thus $|k|=1/\sigma^2=1$ and all the quantities are dimensionless (the physical dimensions can be restored by dimensional analysis at the end of the calculations).

3.1 The Heat Kernel Trace Formula

Let us start with the heat kernel of the Laplace operator acting in H^3. We shall use the method of images. The heat-kernel reads (see, for example, [27, 28])

$$K_t^{H^3}(z,z')=\frac{\exp\left(-t-\frac{d^2(z,z')}{4t}\right)}{(4\pi t)^{\frac{3}{2}}}\frac{d(z,z')}{\sinh d(z,z')}. \tag{3.6}$$

With regard to the heat kernel on $\mathcal{H}^3$, the method of images gives

$$K_t(z,z')=\sum_n \chi^n K_t^{H^3}(z,\gamma_h^n z')=K_t^{H^3}(z,z')+\sum_{n\neq 0}\chi^n K_t^{H^3}(z,\gamma_h^n z')\chi^n, \tag{3.7}$$

where the separation between the identity and the non-trivial periodic geodesic contribution has been done. In our case, the volume $V(F_3)$ of the fundamendal domain F_3 is divergent and we must introduce a regularization. The simplest one is to limit the integration in variable θ between $0<\theta<\pi/2-\varepsilon$, with ε suitable. Thus we have

$$V_\varepsilon(F)=\int_1^N\frac{d\rho}{\rho}\int_0^{2\pi}d\varphi\int_0^{\pi/2-\varepsilon}\frac{\sin\theta}{(\cos\theta)^3}d\theta=2\pi^2 r_+(\cot\varepsilon)^2=2\pi^2 r_+\left(\frac{1}{\varepsilon^2}-\frac{2}{3}+\mathcal{O}(\varepsilon)\right). \tag{3.8}$$

We may determine ε choosing

$$\frac{1}{\varepsilon^2}=\frac{R^2}{r_+^2}-\frac{1}{3}. \tag{3.9}$$

Thus

$$V_R(F)=2\pi^2\frac{R^2}{r_+}-2\pi^2 r_+=\int_0^{\beta_H}d\tau\int_0^{2\pi}d\phi\int_{r_+}^R rdr, \tag{3.10}$$

where the cuttof parameter R has been introduced (see Sect. 2 for notation). The integration over the regularized (fundamental) domain of the diagonal part leads to

$$
\begin{aligned}
\mathrm{Tr} e^{-t\Delta_0}(R) \;&\equiv\; \mathrm{Tr}\, K_t(R) \;=\; V_R(F)\frac{e^{-t}}{(4\pi t)^{\frac{3}{2}}} \\
&+\; 2\pi l_0 e^{-t}\sum_{n\neq 0}\frac{\chi^n}{(4\pi t)^{\frac{3}{2}}}\int_0^{\pi/2}\frac{(\sin\theta)d(z,\gamma_h^n z)}{(\cos\theta)^3\sinh[d(z,\gamma_h^n z)]}e^{-\frac{d^2(z,\gamma_h^n z)}{4t}}d\theta\,, \qquad (3.11)
\end{aligned}
$$

where Δ_0 is a scalar Laplacian, $d(z,\gamma_h^n z)=\cosh^{-1}(1+b_n^2\cos^{-2}\theta)$. The integral over θ can be performed by means of change of the integration variable $\theta\to u$ given by $2\sqrt{ut}=\cosh^{-1}(1+b_n^2\cos^{-2}\theta)$. As a consequence, the resulting integral becomes elementary, i.e.

$$
\mathrm{Tr}\, K_t(R)=V_R(F)\frac{e^{-t}}{(4\pi t)^{\frac{3}{2}}}+4\pi l_0\sum_{n=1}^{\infty}\frac{\chi^n e^{-t}}{b_n^2(4\pi t)^{\frac{3}{2}}}\int_{\frac{n^2 l_0^2}{4t}}^{\infty}e^{-tu}du\,, \qquad (3.12)
$$

since $\cosh^{-1}(1+b_n^2)=nl_0$. As a result one obtains

$$
\mathrm{Tr}\, K_t(R)=V_R(F)\frac{e^{-t}}{(4\pi t)^{\frac{3}{2}}}+\frac{l_0}{2}\sum_{n=1}^{\infty}\frac{\chi^n}{(\sinh\frac{nl_0}{2})^2}\frac{e^{-t-\frac{l_0^2 n^2}{4t}}}{(4\pi t)^{\frac{1}{2}}}\,. \qquad (3.13)
$$

Recently, the above heat-kernel trace has also been computed in [29].

3.2 The Explicit Form of the Zeta Function

In the previous subsection we have derived the heat kernel trace formula. For our purpose it is important that Eq. (3.13) looks (formally) as the Selberg trace formula associated with Laplace operator acting in a compact space $\mathcal{H}^3$ (a group Γ is co-compact). This statement is formal enough, nevertheless let us verify it withstanding a common style of presentation.

First of all we may consider a given (regularized) compact Riemannian manifold as conformally equivalent to one of constant scalar curvature. This is known as the Yamabe problem [30]. This problem has been solved for the case of non-positive scalar curvature in Ref. [31]. Furthermore, let $\{\lambda_j\}_{j=0}^{\infty}$ denote the non-zero isolated eigenvalues (appearing the same number of times as its multiplicity) of positive self-adjoint Laplace operator. Let us introduce a suitable analytic function $h(r)$, where $r_j^2=\lambda_j-1$. It can be shown that $h(r)$ is related to the quantity $k(u(z,\gamma z))$ by means of the Selberg transform (see for example [32, 28] and references therein). Let $\hat{h}(p)$ being the Fourier transform of $h(r)$,

$$
\hat{h}(p)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-irp}h(r)dr\,. \qquad (3.14)
$$

For the derivation of the Selberg trace formula, one has to consider the contributions coming from the identity element in Γ and all γ-type conjugacy classes (the metod of images), namely

$$
\mathrm{Tr}\, h(\Delta_0)=\sum_j h(\lambda_j)=\mathcal{C}(I)+\mathcal{C}(H)=V(F_3)k(0)+\sum_{\{\gamma\}}\chi(\gamma)\int_{F_3}k(u(z,\gamma z))d\mu_3\,. \qquad (3.15)
$$

The first term in the r.h.s. of Eq. (3.15) $\mathcal{C}(I)$ is the contribution of the identity element, while $V(F_3)$ is the (finite) volume of the fundamental domain with respect to the Riemannian measure $d\mu_3=dx_1dx_2dyy^{-3}$. Formally for the non-compact manifold $\mathcal{H}^3$, whose fundamental domain is given by Eq. (3.2), one may put $V(F_3)\sim V_R(F)$, where $V_R(F)$ is given by Eq. (3.10).

Let us consider now the hyperbolic (a topologically non-trivial) contribution and show that it is finite. First it reduces to

$$\mathcal{C}(H) = \sum_{\{\gamma\}} \chi(\gamma) \int_{F_3} k(u(z,\gamma z)) d\mu_3 = \sum_{n\neq 0} \chi^n \int_{F_3} k(u(z,\gamma_h^n z)) d\mu_3 \,. \tag{3.16}$$

Noting that $\chi^n = \chi^{-n}$ and

$$u(z,\gamma_h^n z) = \frac{1}{2}\left(\cosh d(z,\gamma_h^n z) - 1\right) = b_n^2(1+\tan^2\theta)\,, \tag{3.17}$$

with $b_n^2 = \sinh^2(\frac{nl_0}{2})$, one has

$$\mathcal{C}(H) = 4\pi l_0 \sum_{n=1}^{\infty} \chi^n \int_0^{\pi/2} \frac{\sin\theta}{(\cos\theta)^3} k\left(b_n^2(1+\tan^2\theta)\right) d\theta = 2\sum_{n=1}^{\infty} \frac{\chi^n}{b_n^2} \int_{b_n^2}^{\infty} k(x)dx \,. \tag{3.18}$$

Recalling the Selberg transform in the 3-dimensional case [32, 28] one gets

$$\begin{aligned} k(0) &= \int_0^{\infty} \frac{r^2}{2\pi^2}\, h(r)\, dr\,, \\ \int_{b_n^2}^{\infty} k(x)dx &= \frac{1}{4\pi} \hat{h}(n2r_+)\,. \end{aligned} \tag{3.19}$$

Thus the final trace formula reads

$$\mathrm{Tr}\, h(\Delta_0)(R) = V_R(F) \int_0^{\infty} \frac{r^2}{2\pi^2}\, h(r)\, dr + l_0 \sum_{n=1}^{\infty} \chi^n \frac{\hat{h}(n2r_+)}{(\sinh\frac{nl_0}{2})^2}\,. \tag{3.20}$$

The trace formula (3.20) is valid for a large class of $h(r)$ function. In particular, choosing $h(r) = e^{-t(r^2+1)}$ in Eq. (3.20), one obtains the result of Eq. (3.13).

Finally the related zeta function can be calculated by means of the Mellin transform

$$\zeta(s|\Delta_0)(R) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}\, \mathrm{Tr}\, K_t(R) dt\,, \tag{3.21}$$

valid for $\mathrm{Re}\, s > 3/2$. A direct computation gives the analytic continuation of the zeta function in neighborhood of point $s = 0$, i.e.

$$\zeta(s|\Delta_0)(R) = V_R(F) \frac{\Gamma(s-\frac{3}{2})}{(4\pi)^{\frac{3}{2}}\Gamma(s)} + \frac{l_0}{\Gamma(s)\Gamma(1-s)} \int_0^{\infty} \left(2t+t^2\right)^{-s} \Psi(2+t)dt\,, \tag{3.22}$$

where the function

$$\Psi(s) = \sum_{n=1}^{\infty} \frac{\chi^n}{(\sinh\frac{nl_0}{2})^2} e^{-(s-1)l_0 n}\,, \tag{3.23}$$

has been introduced.

For transverse 1-forms, there exists a similar trace formula, (see for example, [34]) and we quote here only the results: there is no gap in the spectrum of Laplace operator $\Delta_1^{\perp}$; the Plancherel measure is $(r^2+1)/(2\pi^2)$ and the heat-kernel trace formula reads

$$\mathrm{Tr}\, e^{-t\Delta_1^{\perp}}(R) = Q(t) \left(V_R(F) + +2l_0(4\pi t) \sum_{n=1}^{\infty} \chi^n \frac{e^{-\frac{n^2 l_0^2}{4t}}}{(\sinh\frac{nl_0}{2})^2} \right)\,, \tag{3.24}$$

with

$$Q(t) = \frac{1}{4(\pi t)^{\frac{3}{2}}} + \frac{1}{2\pi^{\frac{3}{2}} t^{\frac{1}{2}}}\,. \tag{3.25}$$

4 The First Quantum Correction to the Entropy of the BTZ Black Hole

The first quantum correction to the Bekenstein-Hawking entropy may be computed within the Euclidean semiclassical approximation [5] and we shall follow this approach in this section. We have to mention that a more sophisticate approach has been proposed in Ref. [33], where the canonical and microcanonical partition function of the black hole in a cavity with suitable boundary conditions has been investigated. This approach has the merit of a more direct physical understanding, and has been applied to anti-de Sitter black holes in [18].

Within the Euclidean approach, making use of the Chern-Simons representation of the 3-dimensional gravity [21, 22], the one-loop approximation gives

$$\ln Z^{(1)} = \ln(T^{1/2}) - I \,, \tag{4.1}$$

where one is dealing with a compact 3-manifold M and the quantum prefactor T is the Ray-Singer torsion associated with M (see a more precise definition below). In our case, we assume the quantum prefactor to be the same, but

$$\ln Z^{(1)} = \frac{1}{2}\ln T - (I_{BTZ} - I_0) - (B_{BTZ} - B_0) \equiv \frac{1}{2}\ln T - I_P \,, \tag{4.2}$$

in which B_{BTZ} is the usual boundary term which depends on the extrinsic curvature at large spatial distance. The total classical action is divergent; the geometry is non-compact and we have introduced the "reference" background $\mathcal{H}^3_0$ at the tree level [24] and the related volume cutoff R and R_0. With this proposal, in Eq. (4.2) the two boundary terms of the classical contribution cancel for large R and the difference of the on-shell Euclidean classical actions gives rise to (see Sect. 2),

$$I_P = I_{BTZ} - I_0 = -\frac{2}{\pi}\left(V(R) - V_0(R)\right) \to -2\pi r_+ = -\ln Z^{(0)} \,. \tag{4.3}$$

Restoring the correct physical dimension in Eq. (4.3), it is easy to show that the on-shell tree-level partition function $Z^{(0)}$, Eq. (4.3), becomes

$$\ln Z^{(0)} = \frac{4\pi^2 r_+}{16\pi G} \,, \tag{4.4}$$

and this leads to the semiclassical Bekenstein-Hawking entropy

$$S^{(0)} = S_H = \left(r_+\frac{\partial}{\partial r_+} + 1\right)\ln Z^{(0)} = \frac{1}{4}\frac{2\pi r_+}{G} \,. \tag{4.5}$$

So far we have neglected quantum fluctuations. The first on-shell quantum correction of the gravitational quantum fluctuations is given by the square root of the Ray-Singer torsion of the manifold $\mathcal{H}^3$. For a compact hyperbolic manifold the Ray-Singer torsion is the ratio between functional determinants of Laplace operators Δ_k acting on k-forms on $\mathcal{H}^3$ (see for example [21, 22, 34], i.e.

$$T = \frac{\det \Delta_0}{(\det \Delta_1^{\perp})^{1/2}} \,. \tag{4.6}$$

However, in our case a manifold is non-compact and a volume regularization previously introduced will be used. Thus, we have

$$\ln Z^{(1)} = \frac{1}{2}\ln\det\Delta_0 - \frac{1}{4}\ln\det\Delta_1^{\perp} - I_P \,. \tag{4.7}$$

For the tree level term it is necessary to introduce the bare quantity G_B, since the quantum correction are plagued by the ultraviolet divergences and a renormalization procedure must be used. The functional determinants are then calculated by means of a regularization. We shall use the proper-time regularization (zeta-function regularization gives the same finite part), in order to deal explicitly with the ultraviolet divergences.

In the case of 0-forms, one can compute a functional determinant by means of Eq. (3.13). Thus, we have

$$\begin{aligned}\ln\det\Delta_0 &= -\int_\varepsilon^\infty t^{-1}\,\mathrm{Tr}\,e^{-t\Delta_0}dt = -\frac{V_R}{(4\pi)^{3/2}}\Gamma(-\tfrac{3}{2},\varepsilon)+\sum_{n=1}^\infty\frac{\chi^n}{n(\sinh\frac{nl_0}{2})^2}e^{-l_0 n}\\ &= -\frac{V_R}{(4\pi)^{3/2}}\Gamma(-\tfrac{3}{2},\varepsilon)+\ln\mathcal{Z}_0(2)\,,\end{aligned} \tag{4.8}$$

where $\Gamma(-\frac{3}{2},\varepsilon)$ is the incomplete Gamma function, which has two divergent terms as $\varepsilon\to 0$, namely

$$\Gamma(-\tfrac{3}{2},\varepsilon)=\Gamma(-\tfrac{3}{2})-\frac{1}{4(\pi\varepsilon)^{\frac{3}{2}}}+\frac{1}{(4\pi)^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}}+\mathcal{O}(\varepsilon^{\frac{1}{2}})\,, \tag{4.9}$$

and

$$\ln\mathcal{Z}_0(2)=\sum_{k=1}^\infty k\ln\left(1-\chi e^{-2(k+1)\frac{r_+}{\sigma}}\right)\,. \tag{4.10}$$

In a similar way, using Eq. (3.24), one has

$$\ln\det\Delta_1^\perp=-\int_\varepsilon^\infty t^{-1}\,\mathrm{Tr}\,e^{-t\Delta_1^\perp}dt=-\frac{V_R}{8(4\pi\varepsilon)^{\frac{3}{2}}}+\frac{V_R}{2(4\pi)^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}}-\ln\mathcal{Z}_1(1)\,, \tag{4.11}$$

with

$$\ln\mathcal{Z}_1(1)=\sum_{n=1}^\infty\frac{\chi^n}{(\sinh(n\frac{r_+}{\sigma}))^2}\left(\frac{1}{n}+8n(\tfrac{r_+}{\sigma})^2\right)\,. \tag{4.12}$$

As a result

$$\ln Z^{(1)}=\frac{4\pi^2 r_+}{16\pi G}+g(r_+)-F_\varepsilon\,, \tag{4.13}$$

where

$$g(r_+)=\frac{1}{2}\ln\mathcal{Z}_0(2)+\frac{1}{4}\ln\mathcal{Z}_1(1)\,, \tag{4.14}$$

and

$$F_\varepsilon=V_R\left(\frac{1}{2(4\pi)^{3/2}}\Gamma(-\tfrac{3}{2},\varepsilon)+\frac{1}{32(4\pi\varepsilon)^{\frac{3}{2}}}-\frac{1}{8(4\pi)^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}}\right)\,. \tag{4.15}$$

If we define the renormalized quantity

$$\frac{1}{16\pi G_r}=\frac{1}{16\pi G}+\frac{F_\varepsilon}{4\pi^2 r_+}\,, \tag{4.16}$$

we arrive at

$$\ln Z^{(1)} = \frac{\pi r_+}{4G_r} + g(r_+)\,. \tag{4.17}$$

This renormalized one-loop effective action may be thought to describe an effective classical geometry belonging to the same class of the non rotating BTZ black hole solution. This stems from the results contained in [35], where it has been shown that the constraints for pure gravity have an unique solution. As a consequence, one may introduce a new effective radius by means of

$$\ln Z^{(1)} = \frac{\pi R_+}{4\pi G_r}\,, \tag{4.18}$$

where

$$R_+ = r_+ + \frac{4G_r}{\pi} g(r_+)\,, \tag{4.19}$$

mimicking in this way the back reaction of the quantum gravitational fluctuations. As a consequence, the new entropy is given by an effective Bekenstein-Hawking term, namely

$$S^{(1)} = \frac{1}{4}\frac{2\pi R_+}{G_r}\,. \tag{4.20}$$

One can evaluate the asymptotics of the quantity $g(r_+)$ for $r_+ \to \infty$ and $r_+ \to 0$ and then compute the effective radius. Note that $\ln \mathcal{Z}_1(1)$ and $\ln \mathcal{Z}_0(2)$ are esponentially small for large r_+. Thus

$$R_+ \simeq r_+\,, \tag{4.21}$$

and nothing of interesting is present in this limit.

Making use of the results of the Appendix, for small r_+, one has

$$\begin{aligned} R_+ \;\simeq\;& r_+ + \frac{4G_r}{\pi}\left[\frac{\sigma^2}{16 r_+^2}\left(-\ln\left(\frac{2r_+}{\sigma}\right) + 2\gamma + \Psi(2) - \zeta(3)\right)\right. \\ &+ \left.\frac{\sigma\pi^2}{24 r_+} + \frac{1}{4}\ln\left(\frac{r_+}{\sigma\pi}\right) + \mathcal{O}(r_+)\right]\,. \end{aligned} \tag{4.22}$$

One can see that for r_+ sufficiently small the effective radius becomes larger and positive. This means that R_+ (as a function of r_+) reaches a minimum for suitable r_+^*, solution of the equation

$$\frac{4G_r}{\pi} g'(r_+^*) = -1\,. \tag{4.23}$$

This result is in qualitative agreement with a very recent computation of the off-shell quantum correction to the entropy due to a scalar field in the BTZ background [29] and all the qualitative considerations contained there are also valid for the gravitational case we are dealing with. In particular, it appears that the quantum gravitational corrections could become more and more important as soon as the evaporation process continues and thus they cannot be neglected.

5 Concluding Remarks

In this paper the first quantum correction of the BTZ black hole has been evaluated making use of the appropriate Chern-Simons representation of the 3-dimensional gravity. The quantum prefactor, i.e. the Ray-Singer torsion, has been evaluated by means of the proper-time regularization.

In our computation the one-loop ultraviolet and horizon divergences, generally present in the first quantum correction, have also been found and they have been accounted for by means of the introduction of the standard one-loop renormalization procedure of the Newton constant [8]. The semiclassical Bekenstein-Hawking entropy has been also rederived by the improved Euclidean method suggested in [24].

Our result for quantum corrections differs from the one reported in Ref. [19] and are consistent with the detailed computation of the entropy of scalar fields in a BTZ classical backgrounds given in Ref. [29]. With regard to this, horizons divergences of the entropy for scalar fields in the same background have also been investigated in [36].

Finally, our result, even though obtained in the one-loop approximation may be interpreted with a non violation of the area law, but with an effective radius which is the classical one for large black hole mass, but which shrinks, as soon as the black hole evaporation goes on. This seems to suggest the quantum corrections of the gravitational field become more and more important near the end of the evaporation process. As far as this issue is concerned, we observe that the final effective geometrical configuration is the reference space $\mathcal{H}^3_0$, which admits a naked singularity at the origin. As a consequence, the quantum correction seems to have a tendency to avoid the appearance of the naked singularity, in agreement with the "cosmic censorship" hypothesis.

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6 Appendix

In this Appendix we shall investigate the small $t = 2r_+/\sigma$ asymptotics for the quantity $g(t)$, making use of the standard Mellin transform technique [37]. For the sake of simplicity we put $\chi = 1$. To begin with, we observe that $\ln \mathcal{Z}_0(2)$ may be rewritten as

$$\ln \mathcal{Z}_0(2) = \sum_{n=1}^{\infty} n \ln \left(1 - e^{-tn}\right) - \mathbb{H}(t) \,, \tag{6.1}$$

where $\mathbb{H}(t)$ is the Hardy-Ramanujan modular function, given by

$$\mathbb{H}(t) = \sum_{n=1}^{\infty} \ln \left(1 - e^{-tn}\right) \,. \tag{6.2}$$

It satisfies the functional equation

$$\mathbb{H}(t) = -\frac{\pi^2}{6t} - \frac{1}{2} \ln \left(\frac{t}{2\pi}\right) + \frac{t}{24} + \mathbb{H}\left(\frac{4\pi^2}{t}\right) \,. \tag{6.3}$$

For the first term, the Mellin transform representation gives

$$\sum_{n=1}^{\infty} n \ln \left(1 - e^{-tn}\right) = -\frac{1}{2\pi i} \int_{\mathrm{Re}\, z > 2} t^{-z} \Gamma(z) \zeta(z+1) \zeta(z-1) dz \,. \tag{6.4}$$

Shifting the vertical contour to the left, one has a simple pole at $z = 2$, a double pole at $z = 0$ and simple poles at $z = -2m$, $m = 1, 2, \ldots$. The Residue theorem gives for small t

$$\sum_{n=1}^{\infty} n \ln \left(1 - e^{-tn}\right) = -\frac{\zeta(3)}{t^2} + \zeta(-1) \ln t - \zeta'(-1) + \mathcal{O}(t^2) \,. \tag{6.5}$$

With regard to the quantity $\ln \mathcal{Z}_1(1)$, the same technique gives

$$\begin{aligned} \ln \mathcal{Z}_1(1) &= \sum_{n=1}^{\infty} \frac{1}{(\sinh n\frac{t}{2})^2} \left(\frac{1}{n} + 2nt^2\right) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, z>2} dz t^{-z} \Gamma(z) \zeta(z-1) \Big(\zeta(1+z) + 2t^2 \zeta(z-1)\Big) \\ &= \frac{\zeta(3)}{t^2} - \zeta(-1) \ln t + \zeta'(-1) + \frac{1}{t^2} \left(2\gamma + \Psi(2) - \ln t\right) + \mathcal{O}(t^2) \,. \end{aligned} \tag{6.6}$$

As a result, for small t the asymptotics for the quantity $g(t)$ (see Eq. (4.14)) reads

$$g(t) = \frac{1}{4t^2} \left[-\ln t + 2\gamma + \Psi(2) - \zeta(3)\right] + \frac{\pi^2}{12t} + \frac{1}{4} \ln \left(\frac{t}{2\pi}\right) + \frac{t}{48} + \mathcal{O}(t^2) \,. \tag{6.7}$$

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